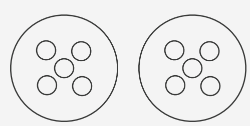
Equivalency

An important component for developing conceptual understandings of fractions is equivalency. Equivalency is much more than multiplying numerator and denominator by a common factor. It involves understanding the ways in which fractional units fit inside of each other. To this end, we will give students many opportunities to work with concrete and pictorial representations.

Math Note

There are two primary models for division that I use when working with fractions. The first is partitioning or partitive division.

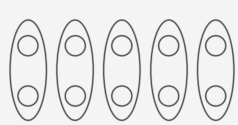
PARTITIVE DIVISION: You know the total, you know the number of groups, and you are trying to find the number in each group. If we view 10 ÷ 2 using partitive division, we know that the total number we have is 10. We are separating it into 2 equal groups to find the number in each group. The related drawing would be,



LANGUAGE: Ten separated into two equal groups.

The other division model we will use is quotitioning or quotitive division. This is also sometimes referred to as measurement division.

QUOTITIVE DIVISION: You know the total, you know the number in each group, and you are trying to find the number of groups. If we view 10 ÷ 2 using quotitive division, we know that the total number we have is 10. We pull off groups of two and get five groups. The related drawing would be,



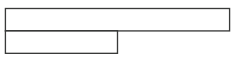
LANGUAGE: How many twos are in ten?

EXAMINING EQUIVALENCY

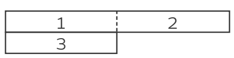
Once students are comfortable with building, drawing, and labeling mixed numbers using words and symbols, I begin to develop the concept of equivalency (the same-sized region can be cut up into smaller pieces). One-half can be cut into two, one-fourth pieces, or three, one-sixth pieces, etc. To do this, continue with mixed numbers and the language of quotitive or measurement division. Using conceptual language gives access to the mathematics that many find difficult.

INVESTIGATION 1

1. Ask students to build $1\frac{1}{2}$ using the fewest number of pieces.

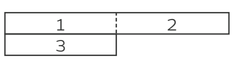


2. **Display and ask**, *How many halves in one and a half?* Once built, students can use a one-half piece to measure the number of halves in $1\frac{1}{2}$.



3. In their journals students record by writing the question, drawing a picture to show the result of measuring by halves, and answering the question.

   How many halves in one and a half?



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4. Have students compare their representations with a neighbor's.

5. As a whole class, invite students to share their representations. Discuss disagreements by inviting students to ask questions to help their peers change their minds.

Math Note

Since many of us want students to label their answers, it is tempting to have students answer "How many halves in one and a half?" as "3 halves." It seems more precise. But three halves is the same as $\frac{3}{2}$ or $1\frac{1}{2}$. That is not the answer to the question. We are answering how many of the indicated unit fractions fit inside the given mixed number. For these problems the answer is a whole number. Later we'll see that the question could be represented symbolically as $1\frac{1}{2} \div \frac{1}{2}$. The answer to this problem is 3, not 3 halves.

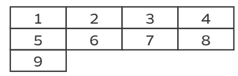
INVESTIGATION 2

1. **Display**
   How many fourths in two and a fourth?

2. Students build a concrete representation using the fewest number of pieces.

3. In their journals, students write the question, draw a picture to show the result of measuring by fourths, and answer the question.

   How many fourths in two and a fourth?



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4. Have students compare their representations with a neighbor's work.

5. As a whole class, invite students to share their representations. Discuss disagreements by inviting students to ask questions to help their peers change their minds.

Do several of this type of problem until students are comfortable with finding the number of times a given unit fraction goes evenly into a mixed number whose fractional part has the same denominator. We'll call these problems "Type 1" problems.

Math Note

A unit fraction is a fraction whose numerator is one.

"TYPE 1" PRACTICE PROBLEMS: For each of these problems students are to first represent using their fraction kit with additional "wholes" if needed (build it). In their journals they write the question, draw a picture to show the result of measuring by the appropriate unit, and write the answer.

1. How many sixths in one and five-sixths?
2. How many fourths in three and three-fourths?
3. How many thirds in two and a third?
4. How many quarters in one and a fourth?
5. How many halves in four?

Add to this list if students need additional practice. Eventually, students will no longer need to first build to be able to draw a picture to represent the answer to the question. That's great. Let them transition at their own pace to no longer building.

EXTENDING TO TYPE 2 PROBLEMS

Once students are comfortable with “Type 1” problems we extend to “Type 2.” For “Type 2” problems, students find the number of times a given unit fraction goes evenly into a mixed number. However, for these problems the denominator of the unit fraction is a multiple of the denominator of the fractional part of the mixed number. For example, “How many eighths in one and a half?” One-eighth is the unit fraction. Its denominator is eight, which is a multiple of two (the denominator of one-half in one and one-half.)

Instructional Note

Before I require students to use symbolic representations, I “nonchalantly” record symbolically as we are debriefing their work. For example, when debriefing “How many fourths in two and a fourth?” I ask how many fourths are in one. They say four. They will see me write “$4/4 = 1$” as I ask, “So four-fourths is equivalent to one?”

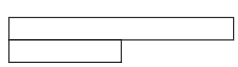
“How many fourths are in two wholes?” (8). “So eight-fourths are in two?” as I write, “$8/4 = 2$”. I also ask, “Eight, one-fourth pieces are in two?” as I write “$8 \times 1/4 = 2$”.

In this way students are introduced to precise symbolic notation before being required to use it. This is a simple way to differentiate. That is, some students will notice and begin using the symbolism before we formally use as a whole class.

INVESTIGATION 3

1 **Display** *How many eighths in one and a half?*

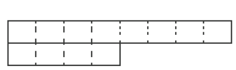
2 Students build one and a half using the fewest number of pieces.



3 Once built, students can show the number of eighths in one and a half.

4 In their journals, have students write the question, draw a picture of the solution, and answer the question.

How many eighths in one and a half?



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5 Have students compare their representations with a neighbor’s.

6 As a whole class, invite students to share their representations. Discuss disagreements by inviting students to ask questions to help their peers change their minds.

Instructional Note

“Type 2” problems allow us to investigate the equivalency of fractions less than one. We will use this same technique to introduce students to the symbolic representation of these equivalencies.

Ask, “How many eighths in one-half?” Students say, “4.” I ask, “Oh, so there are four-eighths in one-half?” as I write $4/8 = 1/2$. “So four-eighths is equivalent to one-half.” I tend to end with “Hmm” or “Isn’t that interesting?” But that’s me.

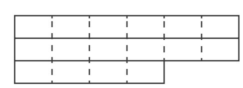
INVESTIGATION 4

1 **Display** *How many sixths in two and two-thirds?*

2 Students first represent using their kit.

3 In their journals, students write the question, draw a picture of the solution, and write the answer.

How many sixths in two and two-thirds?



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4 Have students compare their representations with a neighbor’s.

5 As a whole class, invite students to share their representations. Discuss disagreements by inviting students to ask questions to help their peers change their minds.

We do several “Type 2” problems until students are comfortable drawing the related pictures and answering the question. These experiences with “Type 1” and “Type 2” problems build the foundation for students to understand the standard algorithm they will eventually use when working with equivalent fractions. As students work through these problems they get very good at seeing how the pieces fit inside each other. They will be able to model concretely and pictorially and articulate how and why three one-sixth pieces fit evenly within one-half.

Math Note

At this point we are connecting the conceptual language of division to a concrete and pictorial representation to build an understanding of equivalence. Later we'll introduce the division symbol, connecting to the same use of conceptual language as well as concrete and pictorial representations. That is, they will read $1\frac{1}{2} \div \frac{1}{2}$ as "How many halves in one and a half?"

"TYPE 2" PRACTICE PROBLEMS: For each of these problems students may first build, if needed. In their journals students record by writing the question, drawing a picture of the solutions, and answering the question.

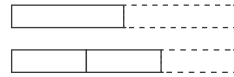
1. How many sixths in three and a half?
2. How many eighths in two and three-fourths?
3. How many fourths in four and a half?
4. How many sixths in three and a third?
5. How many thirds in five?

Instructional Note

Allow the students to decide if they need to first build giving students the opportunity to *choose the appropriate tool*. If they are having difficulties with their drawings or they are inaccurate, I ask them to prove or show using a concrete representation. I then walk away and check back later. I admit that I do keep an eye on them to see if they change their mind or modify their drawings.

Add to this list if students need additional work. If students can accurately draw pictorial representations without a concrete representation, great. We want them to transition to pictorial representations without first needing to build. That is why I keep the kits in a basket. Students can choose when they need the tool (fraction kit). We can also extend beyond the fractions in the kit. For problems with denominators other than 2, 3, 4, 6, and 8 we would connect verbal and pictorial representations (words and drawings).

At this point we have not focused on equivalencies with fractions less than one. I find it more difficult to draw the pictures for fractions less than one. One of the key ideas is that a fraction has meaning when viewed in relation to its whole. Therefore, when drawing pictures of one-half or two-thirds we'd like to see the relationship to the whole.



one-half or $\frac{1}{2}$

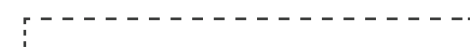
two-thirds or $\frac{2}{3}$

I use the dashed line to represent the remainder of the whole. For many students it's difficult to draw that extra piece. I recommend having them draw the whole using a dashed line. They then trace over the pieces that represent the related quantity.

INVESTIGATION 5

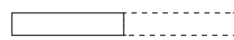
Introduce students to the technique of using a dashed line for representing the whole when drawing fractions less than one.

1. **Display** *One-half*.
2. Have students represent the whole by drawing a dashed line.



3. Have students show one-half of the whole by "cutting" the whole into two equal pieces with a solid line.

4. Students then trace over part of the dashed line to represent one-half.



5. Have students represent two-thirds, one-fourth, and five-sixths using this technique.
6. **Display** *How many eighths in one-half?*
7. In their journals, students write the question, draw a picture of the solution, and write the answer.

   How many eighths in one-half?

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8. Have students compare their representations with a neighbor's work.
9. As a whole class, invite students to share their representations. Discuss disagreements by inviting students to ask questions to help their peers change their minds.

PRACTICE PROBLEMS: For each of these problems students may first build, if needed. In their journals students record by writing the question, drawing a picture of the solution, and answering the question.

1. How many fourths in a half?
2. How many sixths in two-thirds?
3. How many eighths in three-fourths?
4. How many sixths in a third?
5. How many eighths in one-fourth?

RELATED MATHEMATICAL PRACTICES

- Students are making sense of problems as they answer the equivalency questions.
- Students are modeling in mathematics as they use a variety of representations to show equivalency relationships.
- Students begin to examine the structure of mathematics as they observe the teacher "nonchalantly" use symbolic notation to represent equivalency.
- As students play the games, they are constructing viable arguments and critiquing the reasoning of others.

REINFORCING ACTIVITIES AND GAMES

CARD SET: Investigating Equivalencies set (pp. 132–134), one set for each group.

Students can play the following games (for game directions go to page 4):

- *Why? Or Why Not?*
- *All or Nothing*

The question cards in the set can be used for additional practice. Students would record in their journals, as done in the previous practice problem sets.